VII. *On the Theory of the Moon.* By JOHN WILLIAM LUBBOCK, *Esq. V.P. and Treas. R.S.*

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M. POISSON having lately published a very important memoir on the Theory of the Moon, I am induced again to lay before the Society some remarks on this subject.

In this memoir M. POISSON expresses the three coordinates of the moon, namely, her true longitude, her distance, and her true latitude, in terms of the time. The reasons which he adduces for so doing are the same which led me also to deviate from the course which had always been pursued by mathematicians up to the time I commenced the investigation, and which consisted in employing the equations in which the true longitude is the independent variable.

Instead, however, of integrating the equations of motion by the method of indeterminate coefficients, as I have proposed, M. POISSON recommends the adoption of the method of the variation of the elliptic constants. Having reflected much upon this question before I entered upon the investigation, I will venture now to state the reasons which determined me not to employ the latter method.

It seems, in the first place, desirable to introduce uniformity in the methods employed in the theories of the perturbations of the moon, and of the planets, as far as this can be done without the sacrifice of any facility in the solution of the problem. It is not probable, however, that the tables of the planets will be deduced from the variations of their elements. In fact, as I have shown in a former paper, although the results obtained by either method are identical (as is also obvious *à priori*), it is only by numerous reductions that those deduced from the one method are convertible into those deduced from the other. Moreover, in order to obtain, through the variations of the elliptic elements, the inequalities of any given order in the coordinates, the development of the disturbing function must be carried one step further; so that, for example, in the theory of the moon, to obtain all the inequalities depending upon the fourth power of the moon's eccentricity, it would be necessary to obtain the terms depending upon the fifth power of the same eccentricity in the development of the disturbing function.

In the theory of the moon it is necessary to develop many terms in the disturbing function depending on the square of the disturbing mass, and even some depending on the cube; or, in other words, it is, as is well known, insufficient to substitute, in the disturbing function, the elliptic values of the moon's coordinates. The variation of the disturbing function R may be obtained, according to my method, by substituting in the disturbing function the values of the moon's coordinates obtained by a second approximation, or, as in the method of M. POISSON, by substituting in the disturbing function the variations of the elliptic elements due to the disturbing force.

In the former case,

$$\delta R = r \frac{\mathrm{d} R}{\mathrm{d} r} \frac{\delta r}{r} + \frac{\mathrm{d} R}{\mathrm{d} \lambda} \delta \lambda + \frac{\mathrm{d} R}{\mathrm{d} s} \delta s.$$

In the latter,

$$\delta R = \frac{\mathrm{d} R}{\mathrm{d} r} \delta a + \frac{\mathrm{d} R}{\mathrm{d} e} \delta e + \frac{\mathrm{d} R}{\mathrm{d} \varpi} \delta \varpi + \frac{\mathrm{d} R}{\mathrm{d} \varepsilon} \delta \varepsilon + \frac{\mathrm{d} R}{\mathrm{d} \gamma} \delta \gamma + \frac{\mathrm{d} R}{\mathrm{d} \nu} \delta \nu + \frac{\mathrm{d} R}{\mathrm{d} \zeta} \delta \zeta,$$

ζ being equal to $\int n\, d t$.

In the former case it is necessary to multiply 3 series by 3 series, taken two and two; in the latter it is necessary to multiply 7 series by 7 series: and the labour required in the one is to that required in the other about in the same proportion of 3 to 7. The developments of $\delta \frac{\mathrm{d} R}{\mathrm{d} e}$, $\delta \frac{\mathrm{d} R}{\mathrm{d} \varepsilon}$, &c., which have to be separately obtained, require also the same labour.

It is important in a renewed investigation of the lunar theory, considered with a view of improving the lunar tables, to obtain by some independent method the expressions for the coordinates given finally in terms of the mean longitude by MM. DAMOISEAU and PLANA; for when we consider the enormous number of terms which are necessary to be taken into account, and how difficult it is altogether to avoid error in numerical calculations, it is hardly to be expected that their results can be entirely free from error. If, however, the method of the variation of constants be adopted, after the variations of the elements have been obtained, it will require no small labour to effect the necessary substitutions in the elliptic expressions for the coordinates, so as finally to obtain the desired comparison. The quantity of labour necessary in order to bring to conclusion any solution of the problem is a very important consideration, as every additional work, whether in algebra or numbers, brings with it increased danger of mistakes, notwithstanding every care.

The preceding remarks, however, apply particularly to the determination of those inequalities which are not lowered by integration, that is, to almost all those which originate from the terms in R multiplied by $a_{,}^{-3}$; but with respect to others, particularly those selected by M. POISSON, the method which he employs is very preferable.

LAPLACE, in the Mécanique Céleste, vol. iii. p. 171, alludes to an equation of long period of which the argument is twice the longitude of the moon's node, plus the longitude of her perigee, minus three times the longitude of the sun's perigee.

M. POISSON has shown that the coefficient of the corresponding argument in the development of the disturbing function equals zero. I shall now show that this important result may also be arrived at very simply by means of the method of developing R, I gave formerly*.

Employing the same notation,

$$\frac{\mathrm{d}R}{\mathrm{d}\gamma} = r' \frac{\mathrm{d}R}{\mathrm{d}\gamma} \frac{\mathrm{d}r'}{r\,\mathrm{d}\gamma} + \frac{\mathrm{d}R}{\mathrm{d}\lambda'} \frac{\mathrm{d}\lambda'}{\mathrm{d}\gamma} + \frac{\mathrm{d}R}{\mathrm{d}s} \frac{\mathrm{d}s}{\mathrm{d}\gamma}$$

$$R = m_{\prime} \Big\{ - \frac{1}{r_{\prime}} - \frac{r'^2}{4 r_{\prime}^3} \{1 + 3 \cos (2\lambda' - 2\lambda_{\prime}) - 2 s^2\}$$
$$- \frac{r'^3}{8 r_{\prime}^4} \{3 (1 - 4 s^2) \cos (\lambda' - \lambda_{\prime}) + 5 \cos (3\lambda' - 3\lambda_{\prime})\}.$$

It is evident, by mere inspection of the value of R, that the term in question, of which the argument is $3\tau - \xi + 3\xi_{\prime} - 2\eta$, can only arise from the development of

$$- \frac{5 r'^3}{8 r_{\prime}^4} \cos (3\lambda' - 3\lambda_{\prime}),$$

so that we may consider $R = - \frac{5 r'^3}{8 r_{\prime}^4} \cos (3\lambda' - 3\lambda_{\prime})$ in the present investigation, and $\frac{\mathrm{d}R}{\mathrm{d}s} = 0$.

The argument† $3\tau - \xi + 3\xi_{\prime} - 2\eta$, can only be made up, therefore, of the arguments $3\tau - \xi + 3\xi_{\prime}$ and 2η, and $3\tau + 3\xi_{\prime}$ and $\xi + 2\eta$, by subtraction, if we limit ourselves to that part of the coefficient which is multiplied by $e\, e_{\prime}^3 \gamma^2$.

It is therefore necessary to determine the coefficients of R corresponding to the arguments $3\tau + 3\xi_{\prime}$ and $3\tau - \xi + 3\xi_{\prime}$.

By the expression

$$\frac{\mathrm{d}R}{\mathrm{d}e_{\prime}} = r_{\prime} \frac{\mathrm{d}R}{\mathrm{d}r_{\prime}} \frac{\mathrm{d}r_{\prime}}{r_{\prime}\mathrm{d}e_{\prime}} + \frac{\mathrm{d}R}{\mathrm{d}\lambda_{\prime}} \frac{\mathrm{d}\lambda_{\prime}}{\mathrm{d}e_{\prime}},$$

it is evident that the coefficient of $\cos 3\tau + 3\xi_{\prime}$ depends only upon the coefficients of $\cos 3\tau$, $\cos 3\tau + \xi_{\prime}$, and $\cos 3\tau + 2\xi_{\prime}$. On reference to the development of R‡, it will be found, that considering only these terms,

$$R = \frac{m_{\prime} a^3}{a_{\prime}^4} \Big\{ - \underset{[116]}{\frac{5}{8} \cos 3\tau} + \underset{[120]}{\frac{5}{8} e_{\prime} \cos (3\tau + \xi_{\prime})} - \underset{[128]}{\frac{5}{64} e_{\prime}^2 \cos (3\tau + 2\xi_{\prime})} \Big\},$$

and since

$$r_{\prime} \frac{\mathrm{d}R}{\mathrm{d}r} = a_{\prime} \frac{\mathrm{d}R}{\mathrm{d}a_{\prime}}, \text{ and } \frac{\mathrm{d}R}{\mathrm{d}\lambda_{\prime}} = \frac{\mathrm{d}R}{\mathrm{d}\tau},$$

τ being used for $n t - n_{\prime} t$.

$$\frac{\mathrm{d}r_{\prime}}{r_{\prime}\mathrm{d}e_{\prime}} = - \cos \xi_{\prime} - \frac{3}{2} e_{\prime} \cos 2\xi_{\prime} - \frac{17}{8} e_{\prime}^2 \cos 3\xi_{\prime},$$

$$\frac{\mathrm{d}\lambda_{\prime}}{\mathrm{d}e_{\prime}} = 2 \sin \xi_{\prime} + \frac{5}{2} e_{\prime} \sin 2\xi_{\prime} + \frac{13}{4} e_{\prime}^2 \sin 3\xi_{\prime},$$

neglecting terms which are not required.

* Philosophical Transactions, 1832, p. 606.

† I use the letters τ, ξ, $\xi_{\prime}$, and η, where formerly I used t, x, z, and y.

‡ Philosophical Transactions, 1831, p. 266.

3 × numerical coefficient of $\cos(3\tau + 3\xi_{,})$

$$= -4 \times -\frac{5}{8} \times -\frac{17}{8} \times \frac{1}{2} - 4 \times \frac{5}{8} \times -\frac{3}{2} \times \frac{1}{2} - 4 \times -\frac{5}{64} \times -1 \times \frac{1}{2}$$

$$+ 3 \times -\frac{5}{8} \times \frac{13}{4} \times -\frac{1}{2} + 3 \times \frac{5}{8} \times \frac{5}{2} \times -\frac{1}{2} + 3 \times -\frac{5}{64} \times 2 \times -\frac{1}{2}$$

$$= \frac{-340 + 240 - 20 + 390 - 300 + 30}{128} = 0.$$

The first term of the coefficient, therefore, of $\cos(3\tau + 3\xi_{,})$ in the development of the disturbing function equals zero.

It is evident by the expression

$$\frac{dR}{de} = \frac{r\,dR}{dr}\frac{dr}{r\,de} + \frac{dR}{d\lambda}\frac{d\lambda}{de},$$

that the coefficient of $\cos(3\tau - \xi + 3\xi_{,})$ depends solely upon the coefficient of $\cos(3\tau + 3\xi_{,})$; and as this equals zero, the other must also equal zero; and as the coefficient of $\cos(3\tau - \xi + 3\xi_{,} - 2\eta)$ depends solely upon these two coefficients, it must also equal zero, which was the point to be ascertained.

M. Poisson has shown in the memoir before referred to, that the first term in the corresponding inequality of longitude depends only upon this coefficient in the development of R; the inequality is therefore insensible.

It follows equally that the coefficients of all arguments which result from any combination of $3\tau + 3\xi_{,}$ with any multiples of ξ and 2η are also equal to zero.

Note.—The expressions for $\frac{dr'}{r'd\gamma}$ and $\frac{d\lambda'}{d\gamma}$ I gave*, should be as follows:

$$\frac{dr'}{r'd\gamma} = -\frac{\gamma}{2} + \frac{\gamma}{2}(1 - 4e^2)\cos 2\eta - \gamma e \cos(\xi - 2\eta) + \gamma e \cos(\xi + 2\eta)$$

[62] [65] [66]

$$+ \frac{3}{8}\gamma e^2 \cos(2\xi - 2\eta) - \frac{13}{8}\gamma e^2 \cos(2\xi + 2\eta)$$

[77] [78]

$$\frac{d\lambda'}{d\gamma} = -\frac{\gamma}{2}(1 - 4e^2)\sin 2\eta + \gamma e \sin(\xi - 2\eta) - \gamma e \sin(\xi + 2\eta)$$

[62] [65] [66]

$$+ \frac{3}{8} e^2 \sin(2\xi - 2\eta) - \frac{13}{8}\gamma e^2 \sin(2\xi + 2\eta).$$

[77] [78]

The coefficient of the first term (argument 146) in the expression for $\frac{dR}{ds}$†, should be $\frac{408}{137}$ instead of $\frac{204}{137}$.

* Philosophical Transactions, 1832, p. 606. † Ibid. p. 6.